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MATHIEU FUNCTIONS OF GENERAL ORDER: CONNECTION FORMULAE, BASE FUNCTIONS AND ASYMPTOTIC FORMULAE

II. CONNECTION FORMULAE AND BASE FUNCTIONS

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CONTENTS

Connection formulae are examined which relate a solution $y(z)$ of the Mathieu equation $y''+(\lambda+2h^2\cos 2z)\,y = 0$ with the solutions $y(\pm z\pm n\pi)$ generated from it by the symmetry group of the equation. The treatment is exact, and is made first in the context of more general periodic differential equations; the results are then specialized to the Mathieu equation, a function of the third kind, characterized by its asymptotic behaviour as $z\to\infty\mathrm{i}$, being taken as fundamental.

Two parameter ranges are then distinguished, corresponding to the regions of the stability diagram (*a*) where the solutions are always unstable and (*b*) where subregions of stability and instability alternate. Auxiliary parameters are defined in the two cases, and pairs of real-variable base-functions are constructed, appropriate to the ordinary Mathieu equation and to two types of modified equation. These pairs satisfy criteria introduced by Miller (1950).

Comprehensive formulae are derived, relating these base-functions to standard types of Mathieu function, and special attention is given to periodic solutions.

1. Notation and basic properties

The most convenient accepted form of the Mathieu equation for the purposes of this paper

$$y'' + (\lambda - 2q \cos 2x)\, y = 0; \tag{1.1}$$

the parameters λ, q are taken to be real and the case $q = 0$ is specifically excluded from consideration. This equation will be written in two different forms:

$$\text{if} \quad q < 0, \quad y'' + (\lambda + 2h^2 \cos 2x)\, y = 0; \tag{1.2a}$$

$$\text{if} \quad q > 0, \quad y'' + (\lambda - 2h^2 \cos 2x)\, y = 0, \tag{1.2b}$$

where $h^2 = |q|$, h is real and $h > 0$. If $y(x)$ is a solution of (1.2*a*) then $y(x + \frac{1}{2}\pi)$ is a solution of (1.2*b*), and conversely. The forms of the modified Mathieu equation corresponding to (1.2*a*, *b*) are:

$$\text{if} \quad q < 0, \quad y'' - (\lambda + 2h^2 \cosh 2x)\, y = 0; \tag{1.3a}$$

$$\text{if} \quad q > 0, \quad y'' - (\lambda - 2h^2 \cosh 2x)\, y = 0. \tag{1.3b}$$

If $y(x)$ is a solution of (1.2*a*) or of (1.2*b*) then $y(\mathrm{i}x)$ is a solution of (1.3*a*) or of (1.3*b*) respectively, and conversely.

These four forms will be used when the object is to study real-variable solutions of the ordinary and modified Mathieu equations; however, complex-valued solutions with a real independent variable are used, and it is sometimes convenient to consider the extension of a real-variable solution to the complex plane, when the variable x will be treated as complex. When treating the variables as complex, the equation will be written

$$y'' + (\lambda + 2h^2 \cos 2z)\, y = 0, \tag{1.4}$$

the independent variable being denoted by z. It is not necessary to distinguish the four forms (1.2*a*, *b*), (1.3*a*, *b*) since the solutions of any one can be derived from those of (1.4).

(*a*) *Some known properties*

(i) If $y(z)$ is a solution of (1.4), then $\forall n \in \mathbf{Z}$, $y(n\pi \pm z)$ are also solutions.

(ii) There exist two characteristic exponents $\pm\nu = \mp \mathrm{i}\mu$, with corresponding characteristic values $\mathrm{e}^{\pm\pi\mu} = \mathrm{e}^{\pm\mathrm{i}\pi\nu}$ and characteristic, or Floquet, solutions $y(z)$ having the property that $y(z+\pi) = \mathrm{e}^{\pm\pi\mu} y(z)$; every solution of (1.4) satisfies the relation

$$y(z+\pi) + y(z-\pi) = 2 \cosh(\pi\mu)\, y(z). \tag{1.5}$$

The exponents are in general complex, but the characteristic values are either real or complex conjugates, so that $\cosh(\pi\mu)$ is real; it is otherwise unrestricted. The convention will be adopted that $\mathrm{e}^{\pi\mu}$ represents the characteristic value with the larger absolute value, that is, that $\mathrm{Re}(\pi\mu) \geqslant 0$; for $\mathrm{Im}(\pi\mu)$ the normal convention is that it is a continuous function of the parameters, monotone in λ, zero when $\lambda < a_0(q)$ (see (iii) below) and, if it is not zero, having the sign of q; recall that $q \neq 0$.

(iii) There exists a solution of (1.1) having 2π as a period if and only if the characteristic values are equal, when their common value is $e = \pm 1$; there is then one independent characteristic solution, which has the property $y(x+\pi) = ey(x)$ and is either even or odd.

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The values of the parameters for which this situation arises are given by two sequences of characteristic values of λ, functions of q:

$$a_0, a_1, a_2, \ldots; \quad b_1, b_2, \ldots,$$

with $a_n < a_{n+1}$ and, if $q > 0$, $a_n < b_n < a_{n+1}$. Replacing q by $-q$ leaves these values unchanged except that a_n, b_n are interchanged if n is odd. They are characterized by the property that if $\lambda = a_n$ or $\lambda = b_n$ there is an even solution $\mathrm{ce}_n(x)$ or an odd solution $\mathrm{se}_n(x)$ respectively, with characteristic value $(-1)^n$ and with n zeros in $[0, \pi)$. These solutions are the Mathieu functions of integer order n of the first kind.

If $\lambda < a_0$, the characteristic values $e^{\pm\pi\mu}$ are real, positive and distinct. We thus have the following result, important in later considerations:

Lemma 1. $\lambda < a_0(q) \Rightarrow \cosh \pi\mu > 1.$

Also, $a_0 > -2h^2$.

(b) A property of the modified equation (1.3*a*)

For this equation there is an eigenvalue problem for the parameter λ which is similar to the problem for the equation (1.1) whose solution is noted above, and which is considered in Hansen (1962); the corresponding problem for the Lamé equation also appears in the literature and gives rise to Lamé–Wangerin functions. In the following account detailed proofs are not given; they depend on Sturm's theory.

In (1.3*a*), the coefficient of y is negative if $|x|$ is large, and tends to $-\infty$ as $|x| \to \infty$. It follows that there is a solution $y_1(z)$ of (1.4) such that the corresponding solution $y_1(\mathrm{i}x)$ of (1.3*a*) is real, tends to zero as $x \to \infty$ and is positive for large positive x; also, every real solution of (1.3*a*) independent of $y_1(\mathrm{i}x)$ tends to ∞ or to $-\infty$ as $x \to \infty$. If $\lambda \geqslant -2h^2$, the coefficient of y in (1.3*a*) is nowhere positive, whence it follows that no real solution has more than one zero, and that $y_1(\mathrm{i}x)$ has no finite zeros and tends to ∞ as $x \to -\infty$. However, if $\lambda < -2h^2$, the coefficient is positive in the range $|x| < a$, where a is defined by $\lambda + 2h^2 \cosh 2a = 0$, and it can be shown that for given h, every solution oscillates arbitrarily many times in this interval if $|\lambda|$ is sufficiently large. This leads to the following characteristic-value property for λ:

Theorem 1. *There is a sequence*

$$c_0, c_1, c_2, \ldots$$

of values of λ, *functions of* $q = -2h^2$, *such that:*

(i) $c_0 < -2h^2$ *and* $c_{n+1} < c_n$;
(ii) $y_1(\mathrm{i}x) \to 0$ *as* $x \to -\infty$;
(iii) $y_1(\mathrm{i}x)$ *has* n *zeros*;
(iv) $y_1(0) = 0$ *if* n *is odd,* $y_1'(0) = 0$ *if* n *is even*;
(v) $y_1(-\mathrm{i}x) = (-1)^n y_1(\mathrm{i}x)$.

The sequence is characterized by any one of (ii), (iv) *or* (v), *together with* (iii).

For all other values of λ, $y_1(\mathrm{i}x) \to \pm\infty$ *as* $x \to -\infty$.

As a consequence of there follows:

Lemma 2. *If* $\lambda > c_0(q)$, $y_1(\mathrm{i}x) \to \infty$ *as* $x \to -\infty$, $y_1(0) > 0$, $\mathrm{i}y'(0) < 0$ *and* $y_1(\pm\mathrm{i}x)$ *are independent.*

Lemmas 1 and 2 later play similar roles in different contexts.

If $q > 0$, it is convenient to define $c_n(q) = c_n(-q)$.

(*c*) *Nomenclature for solutions*

This is set out in table 1 (II, § 4.1). The notation does not make reference to the characteristic exponent, and the dependence of the functions on λ and on $|q|$ is implicit; the normalizations used are non-standard, being chosen to simplify dependence on the fundamental solution $y_1(z)$ (equation (4.1.1)). However, the asymptotic formulae of parts IV and V enable the relations with other normalizations to be determined approximately. The even and odd functions, both ordinary and modified, are for convenience defined in such a way that at $x = 0$ their values and derivatives respectively are positive.

2. Connection formulae relating solutions of certain periodic differential equations

In the next section, some general properties of connection formulae relating a particular set of solutions of the Mathieu equation are required. It is, however, no more difficult to derive these properties for a wider class of periodic equations.

(*a*) Consider therefore the equation

$$y'' + q(x)\, y = 0 \tag{2.1}$$

with independent and dependent variables x, y, and coefficient $q(x)$ with period π. The variables may be either real or complex; in the former case $q(x)$ is to be continuous on $\boldsymbol{R}$ and in the latter case it is to be analytic on a simply connected open domain $\boldsymbol{D} \subset \boldsymbol{C}$ such that

$$x \in \boldsymbol{D} \Rightarrow -x \in \boldsymbol{D} \quad \textit{and} \quad \forall n \in \boldsymbol{Z}, \quad n\pi + x \in \boldsymbol{D}.$$

In the real-variable case, the solutions of (2.1) are twice differentiable on $\boldsymbol{R}$; in the complex-variable case they will be treated as restricted to $\boldsymbol{D}$, and are then single-valued analytic functions. For convenience the symbol $\boldsymbol{D}$ will be understood to refer to $\boldsymbol{R}$ in the real-variable case.

If $y(x)$ is a solution of (2.1) then $\forall n \in \boldsymbol{Z}$, $y(n\pi + x)$ is also a solution; note also that a non-trivial solution $y(x)$ is a characteristic solution with characteristic exponent $\nu = -\mathrm{i}\mu$ if $y(x+\pi) = \mathrm{e}^{\pi\mu} y(x)$.

Now let $y_i(x)$ $(i = 1, 2)$ be independent solutions of (2.1), forming a basis for the vector space of solutions. Then there are relations with constant coefficients, valid on $\boldsymbol{D}$:

$$\left.\begin{aligned} y_1(x) &= c_{11} y_1(x-\pi) + c_{12} y_2(x-\pi), \\ y_2(x) &= c_{21} y_1(x-\pi) + c_{22} y_2(x-\pi). \end{aligned}\right\} \tag{2.2}$$

Further, the characteristic values are the two eigenvalues of the matrix

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

and the characteristic solutions are the corresponding eigenvectors in the solution space.

Since the Wronskian of any two independent solutions of (2.1), and in particular of two independent characteristic solutions, is constant, it follows readily that the product of the two characteristic values is unity, so they may be written $\mathrm{e}^{\pm\pi\mu}$, and

$$\det \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = 1. \tag{2.3}$$

Also, every solution of (2.1) satisfies (1.5).

(*b*) This appears to be the only general constraint on the coefficients in (2.2). However, any additional symmetry property of the differential equation may permit a choice of basis which does impose such constraints.

In particular, Mathieu's equation, whether with real or complex parameters, and more generally Lamé's and Hill's equations, all have the property that $q(x)$ is an even function. In this case, $y(-x)$ is a solution whenever $y(x)$ is a solution. Let $y_1(x)$ be any solution that is neither even nor odd; then $y_1(x)$, $y_1(-x)$ are independent solutions. On choosing this pair as basis, (2.2) becomes

$$\left.\begin{aligned} y_1(x) &= c_{11}y_1(x-\pi)+c_{12}y_1(\pi-x), \\ y_1(-x) &= c_{21}y_1(x-\pi)+c_{22}y_1(\pi-x), \end{aligned}\right\} \tag{2.4a}$$

and substituting $\pi-x$ for x in this equation gives

$$\left.\begin{aligned} y_1(\pi-x) &= c_{11}y_1(-x)+c_{12}y_1(x), \\ y_1(x-\pi) &= c_{21}y_1(-x)+c_{22}y_1(x). \end{aligned}\right\} \tag{2.4b}$$

From these two pairs of relations it follows that

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad \begin{pmatrix} c_{22} & c_{21} \\ c_{12} & c_{11} \end{pmatrix}$$

are inverse matrices; from this and (2.3),

$$c_{11}c_{22}-c_{21}c_{12} = 1, \quad c_{12}+c_{21} = 0. \tag{2.5}$$

If it is further supposed that $y_1(x)$ is not a characteristic solution, so that $y_1(x)$, $y_1(x-\pi)$ are independent, then $c_{12} \neq 0$ and (2.4*b*) with $-x$ substituted for x in the first of the pair may be rearranged in the following form, more convenient in the applications:

$$\left.\begin{aligned} y_1(-x) &= \beta_+y_1(x)-\gamma y_1(x+\pi), \\ y_1(-x) &= \beta_-y_1(x)+\gamma y_1(x-\pi), \end{aligned}\right\} \tag{2.6}$$

where $\beta_+ = -c_{11}/c_{12}$, $\beta_- = -c_{22}/c_{21}$, $\gamma = c_{21}^{-1} = -c_{12}^{-1}$. The first of (2.5) now becomes

$$\beta_+\beta_-+\gamma^2 = 1; \tag{2.7}$$

subtracting the two equations (2.6) and comparing with (1.5) gives

$$2\gamma \cosh(\pi\mu) = \beta_+-\beta_-, \tag{2.8}$$

whence by eliminating $y_1(x)$ between equations (2.6), we have

$$\beta_+y_1(x-\pi)+\beta_-y_1(x+\pi) = 2\cosh(\pi\mu)\, y_1(-x), \tag{2.9}$$

a companion to (1.5).

The above relations (2.6) and (2.9), and identities (2.7), (2.8), require that $y_1(x)$ shall not be characteristic, but they remain valid if $y_1(x)$, $y_1(-x)$ are linearly dependent. For in this case, $y_1(-x) = ey_1(x)$ where $e = \pm 1$, and the relations and identities are all satisfied with $\beta_+ = \beta_- = e$ and $\gamma = 0$.

(*c*) In the real-variable case, or in the complex-variable case if $q(x)$ and $y_1(x)$ are real for real x, the coefficients β_+, β_-, γ are of course real. However, in the complex-variable case, if $q(x)$ is real for real x as well as being even, there are solutions that are real on the *imaginary* axis, and this leads to a more useful specialization.

The coefficient $q(\mathrm{i}x)$ in the modified equation

$$y'' - q(\mathrm{i}x)\, y = 0 \tag{2.10}$$

corresponding to (2.1) is then also real, and $y(\mathrm{i}x)$ is a solution of (2.10) whenever $y(x)$ is a solution of (2.1), and conversely. Accordingly let $y_1(x)$ of §2 (*b*) above be such a solution; then it is easily seen that $y_1(\mathrm{i}x \pm \pi)$ are conjugate analytic functions, solutions of (2.10). It follows that in (2.6), $\beta_+ = \bar{\beta}_-$ and $\gamma = -\bar{\gamma}$, so that (2.6) take the form given in (4.1.4) in the tables at the end of this part, where $\hat{\beta}$ is real, while (2.7), (2.8) reduce to the form (4.1.5). In addition, (1.5) gives (4.1.2), and (2.9) reduces to (4.1.3).

In general, either the pair (4.1.2), (4.1.3) or the pair (4.1.4) suffices to determine the relation between any three of the solutions $y_1(\pm x + n\pi)$. However, if $\beta = \bar{\beta}$, neither pair is independent and $\beta = \bar{\beta} = \pm 1$, $\hat{\beta} = 0$; nevertheless, it suffices in every case to take one relation from each pair.

It can in fact be shown that the coefficients β, $\hat{\beta}$ take every pair of values satisfying (4.1.5) for some choice of $y_1(x)$; this choice is unique up to a real factor unless $\cosh(\pi\mu) = 0$, which is the case when all solutions of (2.1) have period 4π. There is therefore a sense in which the structure of a system of solutions $y_1(\pm x + n\pi)$ depends only on the characteristic exponents. However, in the application to Mathieu's equation and also to other types of periodic differential equation, the solution $y_1(x)$ will be characterized by a suitably specified asymptotic property, and the coefficients β, $\hat{\beta}$, as well as the exponent $\pi\mu$, will then be well defined functions of the parameters in the differential equation.

3. Application to Mathieu functions

3.1. *Complex variable*

The coefficient in the Mathieu equation is an entire function, so the region $\boldsymbol{D}$ of §2 is the complete complex plane $\boldsymbol{C}$. The equation will first be treated in the form (1.4) and has a solution $y_1(z)$ characterized by

$$y_1(z) \sim (\cos z)^{-\frac{1}{2}}\, \mathrm{e}^{-2h \cos z} \tag{3.1}$$

as $z \to \infty\mathrm{i}$, the square root being real and positive on the imaginary axis; this formula remains valid if $\operatorname{Im} z \to \infty$ with $|\operatorname{Re} z| \leqslant \frac{3}{2}\pi - \delta$ $(\delta > 0)$. This property is derived in part IV independently of the results of the preceding sections; it is, besides, a known result (Meixner & Schäfke 1954). A simple change of variable gives

$$y_1(z-\pi) \sim -\mathrm{i}(\cos z)^{-\frac{1}{2}}\, \mathrm{e}^{2h \cos z} \tag{3.2}$$

as $z \to \infty\mathrm{i}$, whence it is clear that $y_1(z)$ and $y_1(z-\pi)$ are independent. Since $y_1(z)$ is evidently real on the imaginary axis, it will be chosen to play the part of the solution $y_1(x)$ of §2 (*c*), and the solution $y_1(\mathrm{i}x)$ of the modified equation (1.3 *a*) will be identified with the solution of §1 (*b*).

The solution $y_1(z)$ is evidently an analytic function of the real parameters λ, h as well as of z; it follows that the connection coefficients β, $\hat{\beta}$, $\cosh(\pi\mu)$ are also analytic functions of the parameters. Now by (4.1.5), since $\hat{\beta}$ is real, β has no zeros; it follows that $\arg \beta$ has single-valued branches each defined over the entire domain of the parameters, that is $\{h, \lambda \in \boldsymbol{R}; h > 0\}$. At a later stage an appropriate choice of branch will be made.

3.2. *Real variable: generalities*

In the following subsections the auxiliary parameters referred to in the Introduction are defined and pairs of real base-functions are constructed in eight distinct cases. Formulae relating these functions to more familiar types of solution are derived, and also period relations for even and odd functions are expressed in terms of the auxiliary parameters. The eight cases arise from the following three independent choices:

(*a*) two ranges of λ,

(*b*) ordinary or modified equation,

(*c*) $q < 0$ or $q > 0$,

and the bases are chosen to satisfy the criteria introduced by Miller (1950), as follows.

In every case, an appropriate range for the independent variable x comprises intervals, finite or infinite, on each of which the solutions have one of the following characteristics:

(*a*) they are oscillatory;

(*b*) they have hyperbolic behaviour.

The members of a basis are then chosen so that

(*a*) they differ in phase by $\frac{1}{2}\pi$ on each oscillatory interval,

(*b*) the growth rate of their ratio is a maximum on a hyperbolic interval.

The transition between the two types of interval occurs where the coefficient in the differential equation changes sign.

The various cases, with the range of x stated for each, are as follows.

(*a*) Hyperbolic interval only:

(i) $\lambda > c_0(q)$, modified equation, $q < 0$; $x \in \boldsymbol{R}$;

(ii) $\lambda < a_0(q)$, ordinary equation, $q < 0$ *or* $q > 0$; $x \in \boldsymbol{R}$.

(*b*) Oscillatory interval only:

$$\lambda < a_0(q), \text{ modified equation, } q > 0;\ x \in \boldsymbol{R}.$$

(*c*) Intervals of both types:

(i) $\lambda < a_0(q)$, modified equation, $q < 0$; $x \geqslant 0$; there is one finite oscillatory interval and one infinite hyperbolic interval.

(ii) $\lambda > c_0(q)$, modified equation, $q > 0$; $x \in \boldsymbol{R}$; there are two infinite oscillatory intervals separated by a finite hyperbolic interval.

(iii) $\lambda > c_0(q)$, ordinary equation; $q < 0$ and $x \in [0, \pi]$ *or* $q > 0$ and $|x| \leqslant \frac{1}{2}\pi$, there being a simple transformation between these two cases; there are again two oscillatory intervals separated by one hyperbolic interval.

In each of (*c* ii) and (*c* iii) the substitution which preserves the form of the differential equation and reverses the sense of the prescribed range of x has the effect of interchanging the two base-functions. In these cases also, the hyperbolic interval disappears over certain subranges of λ, but the definitions adopted prove to be acceptable on the full range. Several features indicated above appear also for parabolic cylinder functions (Miller 1952).

3.3. *Real variable: the case* $\lambda > c_0(q)$, $q < 0$

(*a*) First, $y_1(\mathrm{i}x) \to 0$ as $x \to \infty$, and by lemma 2 it has no zeros and tends to ∞ as $x \to -\infty$; thus (4.3.4) ($\{y_1(\pm \mathrm{i}x)\}$) forms a suitable basis for the modified equation. Further, by (4.1.4) and lemma 2, $\hat{\beta} \neq 0$, while by (3.2), $iy_1(ix-\pi)$ as well as $y_1(-ix)$ tend to ∞ as $x \to \infty$. Hence by (4.1.4), $y_1(-ix) \sim \hat{\beta} y_1(ix-\pi)$ as $x \to \infty$ and

$$\hat{\beta} > 0. \tag{3.3}$$

To motivate the process used to construct the basis for the ordinary equation it is necessary to introduce some descriptive properties of $y_1(z)$ which can be deduced from the complex-variable asymptotic formulae for $y_1(z)$ obtained in part IV. However, those formulae do not depend on the construction which follows; neither does the validity of the analysis in this section depend on those properties.

If $-2h^2 < \lambda < 2h^2$, $y_1(x)$ has exponential behaviour on $(0, \frac{1}{2}\pi)$ except near $x = a$, where $a \in [0, \frac{1}{2}\pi]$ is defined by $\lambda + 2h^2 \cos 2a = 0$; the exponent is purely imaginary in $(0, a)$ and real in $(a, \frac{1}{2}\pi)$. If $\lambda > 2h^2$ the former property holds on $[0, \frac{1}{2}\pi)$. Consider then the solution

$$y_3(x) = \tfrac{1}{2}\{e^{-i\Phi} y_1(x) + e^{i\Phi} y_1(-x)\}, \tag{3.4}$$

where Φ is a real parameter to be specified later. This solution is evidently real when x is real; it is oscillatory on $[0, a)$ (or on $[0, \frac{1}{2}\pi]$) and is hyperbolic on $(a, \frac{1}{2}\pi]$. On the oscillatory interval, the amplitude of oscillation is $|y_1(x)|$, while the phase difference between two such solutions is the difference of the values of Φ. Then

$$y_3(\pi - x) = \tfrac{1}{2}\{e^{-i\Phi} y_1(\pi - x) + e^{i\Phi} y_1(x-\pi)\},$$

and by means of the connection formulae (4.1.4), with $-x$ in place of x in the first, this can be reduced to

$$y_3(\pi - x) = \tfrac{1}{2}i[\hat{\beta}]^{-1}\{(e^{-i\Phi} + \bar{\beta} e^{i\Phi})\, y_1(x) - (e^{i\Phi} + \beta e^{-i\Phi})\, y_1(-x)\},$$

the validity of the calculations depending on the inequality (3.3); this solution is oscillatory on the same interval as $y_3(x)$, with amplitude $[\hat{\beta}]^{-1}\,|1+\beta e^{-2i\Phi}|\;|y_1(x)|$.

The solutions $y_3(x)$, $y_3(\pi-x)$ differ in phase by $\frac{1}{2}\pi$ if the factor $1+\beta e^{-2i\Phi}$ is real, that is, if $2\Phi = \arg \beta + n\pi$ $(n \in \mathbf{Z})$; apart from scalar factors, all choices of Φ evidently give the same pair of functions, the order in which they appear depending on the parity of n. Let

$$\Phi = \tfrac{1}{2} \arg \beta, \tag{3.5}$$

the specification of a branch of $\arg \beta$ being deferred. This choice of Φ also maximizes the amplitude of oscillation of $y_3(\pi - x)$ on $[0, a)$, corresponding to the application of Miller's second criterion. Then, by using (4.1.5),

$$y_3(\pi - x) = \tfrac{1}{2}i\beta^{*}\{e^{-i\Phi} y_1(x) - e^{i\Phi} y_1(-x)\}, \tag{3.6}$$

where $\beta^{*} = (1+|\beta|)/\hat{\beta}$ as defined in (4.3.3). Since $y_1(\pm x)$ are independent solutions, so also are $y_3(x)$, $y_3(\pi-x)$, and the basis for the ordinary equation, as well as the auxiliary parameters, may be defined in accordance with (4.3.6) and (4.3.1) respectively.

The solutions $\frac{1}{2}\{y_3(x) \pm y_3(\pi - x)\}$ are also independent and their derivative or value respectively vanishes at $x = \frac{1}{2}\pi$. It follows that

$$y_3(\tfrac{1}{2}\pi) \neq 0, \quad y_3'(\tfrac{1}{2}\pi) \neq 0, \tag{3.7}$$

and hence by continuity the signs of these quantities are independent of λ, h; this property is required later.

At this point the formulae (4.3.3) relating to the case of integer order can conveniently be derived, and odd and even solutions of the modified and ordinary equations may be defined in accordance with (4.3.5) and (4.3.7). It is desirable that the value of the even functions and the derivative of the odd functions should be positive at $x = 0$; this is easily verified by means of lemma 2. The expressions for ce (x), se (x) in terms of $y_3(x)$, $y_3(\pi - x)$ given in (4.3.7) can be obtained from (3.4) and (3.6), and the period relations (4.3.8), (4.3.9) follow with the aid of the connection formulae (4.1.4) which may be written in the form (4.3.2). The particular values

$$\left.\begin{aligned} \mathrm{ce}\,(\tfrac{1}{2}\pi) &= (\cos\Phi + [\beta^*]^{-1}\sin\Phi)\, y_3(\tfrac{1}{2}\pi), \\ \mathrm{ce}'\,(\tfrac{1}{2}\pi) &= (\cos\Phi - [\beta^*]^{-1}\sin\Phi)\, y_3'(\tfrac{1}{2}\pi) \end{aligned}\right\} \tag{3.8}$$

will be required later.

(*b*) It is now possible to select a suitable branch of arg β and to describe qualitatively its dependence on λ. If $\lambda = a_0(q)$, the characteristic solution is ce (x) and has period factor 1 since it has no zeros. It follows from the period relations (4.3.9) that $e_1 = 1$ and $e_2 = -1$, so that by (4.3.3), $\sin 2\Phi > 0$ and $\cos 2\Phi < 0$. Accordingly, arg β is defined so that

$$\lambda = a_0(q) \Rightarrow 2\Phi = \arg\beta = \pi - \arctan\hat{\beta} \tag{3.9}$$

with $0 < \arctan\hat{\beta} < \frac{1}{2}\pi$; from (4.3.3) it follows that also

$$\Phi = \arctan\beta^*. \tag{3.10}$$

By continuity the definition is extended to all real values of λ, h $(h > 0)$.

For fixed h, as λ increases from the value $c_0(q)$, it can be deduced by Sturm theory that ce′ $(\frac{1}{2}\pi)$ and ce $(\frac{1}{2}\pi)$ vanish and change sign alternately as λ passes through the values a_0, a_1, a_2, ..., and that when $\lambda = a_n$, ce (x) has period factor $e_1 = (-1)^n$, while from (4.3.9) $e_2 = -1$. Hence by (3.8), $\tan\Phi - (-1)^n\beta^*$ changes sign as λ passes through the value a_n and does not vanish for any other values of $\lambda > c_0(q)$. It follows that as λ increases from the value $c_0(q)$, Φ passes through successive values $n\pi \pm \arctan\beta^*$, which lie in successive quadrants, *either* in increasing sequence *or* in decreasing sequence. In fact the former holds; for if $c_0(q) < \lambda < a_0(q)$, then $\cosh(\pi\mu) > 1$ by lemma 1, so that $\sin 2\Phi > \hat{\beta}/|\beta|$, and it follows by continuity from (3.9) that $2\Phi < \pi - \arctan\hat{\beta}$, that is, $\Phi < \arctan\beta^*$. The required conclusion follows; in particular, $\Phi = \pi - \arctan\beta^*$ when $\lambda = a_1(q)$.

Similarly, when $\lambda = b_n(q)$, the characteristic function is se (x) and $e_1 = (-1)^n$, $e_2 = 1$ so that $\tan\Phi = (-1)^n/\beta^*$. Hence the values of Φ for which $\tan\Phi = \pm 1/\beta^*$ are also taken in increasing sequence as λ increases. Table 2 can now be constructed.

If $\lambda = a_0(q)$, then ce $(\frac{1}{2}\pi)$, $\cos\Phi$ and $\sin\Phi$ are all positive, and similarly if $\lambda = a_1(q)$, ce′ $(\frac{1}{2}\pi)$ and $\cos\Phi$ are negative while $\sin\Phi$ is positive. It follows from (3.8) that for all values of h and of $\lambda > c_0(q)$,

$$y_3(\tfrac{1}{2}\pi) > 0, \quad y_3'(\tfrac{1}{2}\pi) > 0, \tag{3.11}$$

since it has already been shown, (3.7), that their signs are constant; this result again is required for use later.

(*c*) To complete the information required for the tables in §4 relating to the case $\lambda > c_0(q)$, $q < 0$, it is necessary to introduce suitable characteristic solutions of the ordinary and modified equations.

If $y(z)$ is any solution of (1.4) not having characteristic value $e^{-\pi\mu}$ then

$$e^{\pi\mu}y(z) - y(z-\pi) \tag{3.12}$$

is a solution with characteristic value $e^{\pi\mu}$; for the ordinary equation (1.2*a*) the solution

$$e^{\pi\mu}y_3(x) - y_3(x-\pi)$$

suggests itself, since it is real when $e^{\pi\mu}$ is real. However, to write this in terms of the chosen basis it is necessary to express $y_3(x-\pi)$ in terms of this basis. One method is to express $y_3(-x)$ in terms of ce $(-x)$, se $(-x)$ and hence in terms of ce (x), se (x), and so finally in terms of the basis; $\pi - x$ is then substituted for x in the result. Introducing for convenience a factor $|\beta|$ gives the formula for me (x) appearing under (4.3.7).

With the convention already adopted in §1(*a*), that $|e^{\pi\mu}| \geqslant 1$, me (x) thus defined cannot vanish identically, for if $e^{\pi\mu}$ is real, sinh $(\pi\mu) \geqslant 0$ and if not, sinh $(\pi\mu)$ is imaginary. The appropriate solution with characteristic value $e^{-\pi\mu}$ is me $(\pi - x)$ whose expression in terms of the basis can be written down immediately.

The modified function Me (x) can be defined so that it is real if $\nu = -i\mu$ is real, and a suitable function is

$$\mathrm{Me}\,(x) = \tfrac{1}{2}i\hat{\beta}\{e^{\pi\mu}y_1(z) - y(z-\pi) - e^{-\pi\mu}y_1(z) + y_1(z+\pi)\},$$

where $z = ix$, and by means of the connection formulae (4.3.2) this is easily reduced to the form given under (4.3.5).

3.4. *Real variable: the case* $\lambda > c_0(q)$, $q > 0$

(*a*) For the ordinary equation (1.2*b*) the basis (4.3.11) and the definitions under (4.3.12) are a natural choice; it follows from (3.11) that ce* (0) and se*′ (0) are positive. The remaining formulae relating to this equation can be derived from results already obtained.

The variable x in the modified equation (1.3*b*) corresponds to the substitution of $z = ix + \frac{1}{2}\pi$ in the complex-variable equation (1.4), and $y_1(z)$ is then exponential in behaviour for real x, the exponent being imaginary if $|x|$ is sufficiently large, to be precise, if $2h^2 \cosh 2x - \lambda > 0$. The solution

$$y_4(x) = \tfrac{1}{2}\{e^{i\theta}y_1(ix + \tfrac{1}{2}\pi) + e^{-i\theta}y_1(ix - \tfrac{1}{2}\pi)\}$$

of (1.3*b*) is real for real x and is oscillatory for sufficiently large $|x|$. The application of Miller's criteria to the pair $y_4(\pm x)$ by a process similar to that used in §3.3(*a*) leads to the assignment $\theta = \frac{1}{4}\pi - \Phi$ and to the definition (4.3.17). Also,

$$y_4(-x) = \tfrac{1}{2}\beta^{\dagger}\{e^{-i(\frac{1}{4}\pi+\Phi)}y_1(ix + \tfrac{1}{2}\pi) + e^{i(\frac{1}{4}\pi+\Phi)}y_1(ix - \tfrac{1}{2}\pi)\}, \tag{3.13}$$

where $\beta^{\dagger} = |\beta| + \hat{\beta} = (|\beta| - \hat{\beta})^{-1} > 1$. The two solutions $y_4(\pm x)$ differ in phase by $\frac{1}{2}\pi$ and $\beta^{\dagger}$ is the maximized ratio of the amplitudes of oscillation when x is large and positive. The remaining definitions and formulae of (4.3.18) and (4.3.19) can be derived without further difficulty.

(*b*) Other standard solutions are Fey_n, Gey_n (McLachlan 1947) and $M_\nu^{(j)}$ (Meixner & Schäfke 1954); these are all solutions of the modified equation (1.3*b*). The former are simply solutions which, when x is large and positive, differ in phase by $\frac{1}{2}\pi$ from the odd or even solution respectively; these functions of the second kind are thus, by (4.3.19), multiples of

$$\mathrm{Re}\,[\hat{\beta}y_1(\tfrac{1}{2}\pi + ix)] \quad \text{if} \quad e_1 = 1, \text{ or } \mathrm{Im}\,[\hat{\beta}y_1(\tfrac{1}{2}\pi + ix)] \quad \text{if} \quad e_1 = -1.$$

The functions $M_\nu^{(3)}$, $M_\nu^{(4)}$ are characterized by the property

$$M_\nu^{(3)}(x) \sim H_\nu^{(1)}(2h\cosh x) \quad (x \to \infty),$$
$$M_\nu^{(4)}(x) \sim H_\nu^{(2)}(2h\cosh x) \quad (x \to \infty),$$

where $H_\nu^{(1)}$, $H_\nu^{(2)}$ are the usual Hankel functions. The first gives

$$M_\nu^{(3)}(x) \sim (\pi h \cosh x)^{-\frac{1}{2}} \, e^{-i\pi(\frac{1}{4}+\frac{1}{2}\nu)} \, e^{2ih\cosh x};$$

by comparing this with

$$y_1(ix+\tfrac{1}{2}\pi) \sim (-i\sinh x)^{-\frac{1}{2}} \, e^{2ih\sinh x} \quad \text{as} \quad x \to \infty,$$

obtained from (3.1), it follows that

$$M_\nu^{(3)}(x) = -i(\pi h)^{-\frac{1}{2}} \, e^{-\frac{1}{2}i\pi\nu} y_1(ix+\tfrac{1}{2}\pi);$$

similarly,

$$M_\nu^{(4)}(x) = i(\pi h)^{-\frac{1}{2}} \, e^{\frac{1}{2}i\pi\nu} y_1(ix-\tfrac{1}{2}\pi).$$

The remaining functions $M_\nu^{(1)}$, $M_\nu^{(2)}$ are expressible in terms of these two (Meixner & Schäfke 1954).

3.5. *Real variable: the case* $\lambda < a_0(q)$

The most convenient solution to take as starting point is now $y_1(z-\pi)$, or the corresponding solution $y_1(ix-\pi)$ of the modified equation (1.3*a*). The behaviour of this last is exponential for real x, with real exponent if $|x| > a$ and imaginary exponent if $|x| < a$, where $a > 0$ is defined by $\lambda + 2h^2 \cosh 2a = 0$. Since $y_1(ix)$, $y_1(ix-\pi)$ are independent and since, by lemma 1, $\cosh(\pi\mu) \neq 0$, it follows from (4.1.2) that the two solutions $y_1(ix \pm \pi)$ are independent; they are evidently complex conjugates for real x. Hence, if c, θ are real, the solution

$$y_2(ix) = \tfrac{1}{2}c\{e^{i\theta} y_1(ix-\pi) + e^{-i\theta} y_1(ix+\pi)\}$$

is real for real x; it is hyperbolic if $|x| > a$ and oscillatory with amplitude $c|y_1(ix-\pi)|$ if $|x| < a$. Now

$$y_1(ix) = \tfrac{1}{2}(\cosh \pi\mu)^{-1} \{y_1(ix-\pi) + y_1(ix+\pi)\}$$

is already of this form and tends to zero as $x \to \infty$, so it is taken as one member of a basis. For the second member, θ is assigned the value $\frac{1}{2}\pi$; a suitable value for the constant c, which should be asymptotically equal to $(\cosh \pi\mu)^{-1}$, is obtained as follows.

The following solutions of (1.4) have characteristic values $e^{\pm\pi\mu}$ respectively:

$$e^{\pm\pi\mu} y_1(z+\pi) - e^{\mp\pi\mu} y_1(z-\pi);$$

they can be written

$$\pm \sinh(\pi\mu) \{y_1(z+\pi) + y_1(z-\pi)\} + \cosh(\pi\mu) \{y_1(z+\pi) - y_1(z-\pi)\}.$$

Thus, by putting $c = (\sinh \pi\mu)^{-1}$ in accordance with (4.2.2), the solutions

$$\begin{aligned} \mathrm{Me}^{\pm}(x) &= y_1(ix) \pm i y_2(ix) \\ &= \pm(\sinh 2\pi\mu)^{-1} \{e^{\pm\pi\mu} y_1(ix+\pi) - e^{\mp\pi\mu} y_1(ix-\pi)\} \end{aligned} \tag{3.14}$$

of (1.3*a*) have characteristic values $e^{\pm\pi\mu}$ respectively.

Next, adding the two formulae (4.1.4) and using the first of (4.1.5) gives

$$y_1(-z) = \mathrm{Re}\,\beta\, y_1(z) + \mathrm{Im}\,\beta \tanh(\pi\mu)\, y_2(z).$$

Now the sum of the squares of the coefficients in this relation is

$$|\beta|^2 - (\mathrm{Im}\,\beta)^2 \operatorname{sech}^2(\pi\mu) = |\beta|^2 - \hat{\beta}^2 = 1;$$

it is thus convenient to define the phase parameter Φ in accordance with (4.2.1). The result of substituting into the preceding formula is

$$y_1(-z) = \cos 2\Phi\, y_1(z) + \sin 2\Phi\, y_2(z); \tag{3.15}$$

obtaining a second relation by substituting $-z$ for z in this, and then eliminating $y_1(-z)$ with the aid of (3.15) gives

$$y_2(-z) = \sin 2\Phi\, y_1(z) - \cos 2\Phi\, y_2(z).$$

By means of these formulae and (3.14), one obtains

$$\mathrm{Me}^{\pm}(-x) = \mathrm{e}^{\pm 2\mathrm{i}\Phi}\,\mathrm{Me}^{\mp}(x);$$

now the two solutions $\mathrm{Me}^{\pm}(\mp \mathrm{i}x)$ of the ordinary equation (1.2*a*), both of characteristic value $\mathrm{e}^{\pi\mu}$, are easily seen to be complex conjugate for real x, so the solution

$$\mathrm{me}\,(x) = \mathrm{e}^{-\mathrm{i}\Phi}\,\mathrm{Me}^{+}(-\mathrm{i}x) = \mathrm{e}^{\mathrm{i}\Phi}\,\mathrm{Me}^{-}(\mathrm{i}x) \qquad (3.16)$$

is real.

With the necessary further definitions, which are all natural ones, the remaining formulae of §4.2 can now be derived, including those relating to the case $\lambda < a_0(q)$, $q > 0$. The only formula which is not altogether straightforward is (4.2.4) relating to the eigenvalue problem considered in §1(*b*). Here, by an argument similar to that of §3.3(*b*), it can be shown that as λ decreases from the value $a_0(q)$ with fixed $q < 0$, 2Φ passes successively through integer multiples of π in *decreasing* sequence, the first to be met being zero. Thus by (3.15) the value $\Phi = -\frac{1}{2}n\pi$ corresponds to the characteristic value $\lambda = c_n(q)$.

It is also necessary to show that ce (0) and se′ (0) are both positive, and similarly for the other pairs of even and odd functions. Consider the ordinary equation (1.2*a*); since $\lambda < a_0(q)$, no real solution can have more than one zero, whence the characteristic solution me (x) can have no zeros. It follows that equally me′ (x) can have no zeros, for otherwise me″ (x) and hence me (x) itself would have zeros. Since the characteristic value $\mathrm{e}^{\pi\mu}$ is greater than unity, it is thus evident that me (x) and me′ (x) have the same constant sign for all x, and for all h and λ $(< a_0(q))$. Now when $\lambda = c_0(q)$, $\Phi = 0$ and by the first of (4.2.3) and of (4.2.6),

$$\mathrm{me}\,(0) = \mathrm{ce}\,(0) = y_1(0) > 0;$$

thus the constant sign is positive. The required conclusion follows for all four pairs of even and odd functions.

It seems pertinent to remark finally that the coefficients in all the connection formulae relating to the case $\lambda < a_0(q)$ are simple functions of Φ, and apart from (4.2.7) and the definition of $y_2(z)$ they do not depend on μ. This is clearly a consequence of the precise definitions of $y_2(z)$ and of Φ, and would seem to justify these definitions in spite of some apparent artificiality, particularly in that of Φ.

4. Tables

4.1. *General*

(*a*) *Forms of the Mathieu and modified Mathieu equations*

The forms used are set out at the beginning of §1.

Fundamental solution of $y'' + (\lambda + 2h^2 \cos 2z)\, y = 0$:

$$y_1(z) \sim (\cos z)^{-\frac{1}{2}}\,\mathrm{e}^{-2h\cos z} \quad (h > 0) \qquad (4.1.1)$$

as $\mathrm{Im}\, z \to \infty$ with $|\mathrm{Re}\, z| \leqslant \frac{3}{2}\pi - \delta \quad (\delta > 0)$.

This characterizes y_1 completely; it is a function of the third kind, recessive as $\mathrm{Im}\, z \to \infty$ with $\mathrm{Re}\, z = 0$;

$$y(\pm z + n\pi) \quad (n \text{ an integer})$$

are also solutions.

Connection formulae:

$$y_1(z-\pi)+y_1(z+\pi) = 2\cosh(\pi\mu)\, y_1(z), \tag{4.1.2}$$

$$\beta y_1(z-\pi)+\bar{\beta} y_1(z+\pi) = 2\cosh(\pi\mu)\, y_1(-z), \tag{4.1.3}$$

$$\left.\begin{aligned} y_1(-z) &= \beta y_1(z) - \mathrm{i}\hat{\beta} y_1(z+\pi), \\ &= \bar{\beta} y_1(z) + \mathrm{i}\hat{\beta} y_1(z-\pi), \end{aligned}\right\} \tag{4.1.4}$$

where $\hat{\beta}$ is real,

and

$$\left.\begin{aligned} \operatorname{Im}\beta &= \hat{\beta}\cosh(\pi\mu) \\ |\beta|^2 &= 1+\hat{\beta}^2. \end{aligned}\right\} \tag{4.1.5}$$

In (4.1.2), $\mu = \mathrm{i}\nu$ where ν is the usual characteristic exponent, for which there exist solutions of the ordinary equation with the property (see §1 (*a*))

$$y(x+\pi) = \mathrm{e}^{\pm \mathrm{i}\pi\nu} y(x) = \mathrm{e}^{\pm\pi\mu} y(x).$$

The formulae (4.1.3), (4.1.5) effectively define the coefficients β, $\hat{\beta}$. The main part of the tables follows; it is divided into two sections, one applicable when $\lambda < a_0(h^2)$ and the other when $\lambda > c_0(h^2)$; here, a_0 is the least value of λ for which there is a periodic solution of the ordinary equation – this is standard notation – while c_0 is the largest value of λ for which the modified equation

$$y'' - (\lambda + 2h^2 \cosh 2x)\, y = 0$$

has an even solution that tends to zero as $x \to \infty$. Since $c_0(q) < -2h^2 < a_0(q)$, there is an overlap of the domains of applicability of the two sections.

Table 1. Nomenclature for solutions

type of solution		notation: ordinary function	notation: modified function
even, odd	$q < 0$	ce (x), se (x)	Ce (x), Se (x)
	$q > 0$	ce* (x), se* (x)	Ce* (x), Se* (x)
characteristic	$q < 0$	me (x)	Me (x), Me$^{\pm}$ (x)
(Floquet)	$q > 0$	me* (x)	Me* (x)

4.2. *Bases and relations*, $\lambda < a_0(q)$

(*a*) *Auxiliary parameters* (μ, Φ)

The parameter μ is real and positive, and Φ satisfies

$$\left.\begin{aligned} \operatorname{Re}\beta &= \cos 2\Phi, \\ \operatorname{Im}\beta &= \coth(\pi\mu)\sin 2\Phi, \\ \hat{\beta} &= \operatorname{cosech}(\pi\mu)\sin 2\Phi, \end{aligned}\right\} \tag{4.2.1}$$

together with the condition

$$\arg|\beta - 2\Phi| < \tfrac{1}{2}\pi.$$

The phase parameter Φ is determined by these relations, which are not independent; 2Φ and $\arg\beta$ lie in the same quadrant and are equal if either is a multiple of $\frac{1}{2}\pi$; also, $-\infty < \Phi < \frac{1}{4}\pi$. The connection formulae (4.1.2), (4.1.3) may be expressed in terms of these parameters; for the determination of $\arg\beta$, see §3.3 above.

(*b*) *Modified functions*, $q < 0$; $y'' - (\lambda + 2h^2 \cosh 2x)\, y = 0$

Basis:
$$\{y_1(\mathrm{i}x),\ y_2(\mathrm{i}x)\}, \tag{4.2.2}$$

where
$$y_2(z) = \tfrac{1}{2}\mathrm{i}\,(\sinh \pi\mu)^{-1}\,\{y_1(z-\pi) - y_1(z+\pi)\}.$$

This pair of solutions satisfies the criteria of Miller (1950); see §3.2.

Definitions and connection formulae:

$$\left.\begin{aligned} \mathrm{Ce}\,(x) &= \cos\Phi\, y_1(\mathrm{i}x) + \sin\Phi\, y_2(\mathrm{i}x),\\ \mathrm{Se}\,(x) &= -\sin\Phi\, y_1(\mathrm{i}x) + \cos\Phi\, y_2(\mathrm{i}x),\\ \mathrm{Me}^{\pm}\,(x) &= y_1(\mathrm{i}x) \pm \mathrm{i}y_2(\mathrm{i}x) = \mathrm{e}^{\pm 2\mathrm{i}\Phi}\,\mathrm{Me}^{\mp}\,(-x),\\ y_1(-\mathrm{i}x) &= \cos 2\Phi\, y_1(\mathrm{i}x) + \sin 2\Phi\, y_2(\mathrm{i}x),\\ y_2(-\mathrm{i}x) &= \sin 2\Phi\, y_1(\mathrm{i}x) - \cos 2\Phi\, y_2(\mathrm{i}x), \end{aligned}\right\} \tag{4.2.3}$$

An eigenvalue problem:

If
$$\left.\begin{aligned} \Phi &= -\tfrac{1}{2}n\pi \quad (n = 0, 1, 2, \ldots),\\ y_1(-\mathrm{i}x) &= (-1)^n\, y_1(\mathrm{i}x), \end{aligned}\right\} \tag{4.2.4}$$
then

so that
$$y_1(\mathrm{i}x) \to 0 \quad \text{as} \quad x \to \pm\infty.$$

(*c*) *Ordinary functions*, $q < 0$; $y'' + (\lambda + 2h^2 \cos 2x)\, y = 0$

Basis:
$$\{\mathrm{me}\,(x),\ \mathrm{me}\,(-x)\}, \tag{4.2.5}$$

where
$$\mathrm{me}\,(x) = \mathrm{e}^{-\mathrm{i}\Phi}\,\mathrm{Me}^{+}\,(-\mathrm{i}x).$$

Definitions and connection formulae:

$$\left.\begin{aligned} &\left.\begin{aligned}\mathrm{ce}\,(x) &= \mathrm{Ce}\,(\mathrm{i}x)\\ \mathrm{se}\,(x) &= -\mathrm{i}\,\mathrm{Se}\,(\mathrm{i}x)\end{aligned}\right\} = \tfrac{1}{2}\{\mathrm{me}\,(x) \pm \mathrm{me}\,(-x)\},\\ &y_1(x) = \tfrac{1}{2}\{\mathrm{e}^{\mathrm{i}\Phi}\,\mathrm{me}\,(x) + \mathrm{e}^{-\mathrm{i}\Phi}\,\mathrm{me}\,(-x)\},\\ &y_2(x) = -\tfrac{1}{2}\mathrm{i}\{\mathrm{e}^{\mathrm{i}\Phi}\,\mathrm{me}\,(x) - \mathrm{e}^{-\mathrm{i}\Phi}\,\mathrm{me}\,(-x)\}. \end{aligned}\right\} \tag{4.2.6}$$

Period relation:
$$\mathrm{me}\,(x+\pi) = \mathrm{e}^{\pi\mu}\,\mathrm{me}\,(x). \tag{4.2.7}$$

(*d*) *Ordinary functions*, $q > 0$; $y'' + (\lambda - 2h^2 \cos 2x)\, y = 0$

Basis:
$$\{\mathrm{me}^*\,(x),\ \mathrm{me}^*\,(-x)\}, \tag{4.2.8}$$

where
$$\mathrm{me}^*\,(x) = \mathrm{me}\,(x + \tfrac{1}{2}\pi).$$

Definitions and connection formulae:

$$\left.\begin{aligned} &\left.\begin{aligned}\mathrm{ce}^*\,(x)\\ \mathrm{se}^*\,(x)\end{aligned}\right\} = \tfrac{1}{2}\{\mathrm{me}^*\,(x) \pm \mathrm{me}^*\,(-x)\},\\ &y_1(x+\tfrac{1}{2}\pi) = \tfrac{1}{2}\{\mathrm{e}^{\mathrm{i}\Phi}\,\mathrm{me}^*\,(x) + \mathrm{e}^{-\pi\mu}\,\mathrm{e}^{-\mathrm{i}\Phi}\,\mathrm{me}^*\,(-x)\},\\ &y_1(-x-\tfrac{1}{2}\pi) = \tfrac{1}{2}\{\mathrm{e}^{-\mathrm{i}\Phi}\,\mathrm{me}^*\,(x) + \mathrm{e}^{-\pi\mu}\,\mathrm{e}^{\mathrm{i}\Phi}\,\mathrm{me}^*\,(-x)\}. \end{aligned}\right\} \tag{4.2.9}$$

(*e*) *Modified functions*, $q > 0$; $y'' - (\lambda - 2h^2 \cosh 2x)\, y = 0$

Basis:
$$\{\mathrm{Ce}^*\,(x),\ \mathrm{Se}^*(x)\}, \tag{4.2.10}$$

where
$$\mathrm{Ce}^*\,(x) = \mathrm{ce}^*\,(\mathrm{i}x),\ \mathrm{Se}^*\,(x) = -\mathrm{i}\,\mathrm{se}^*\,(\mathrm{i}x).$$

Definitions and connection formulae:

$$\left.\begin{aligned} \text{Defining}\quad \mathrm{Me}^*(x) &= \mathrm{me}^*(\mathrm{i}x) = \mathrm{e}^{-\mathrm{i}\Phi}\,\mathrm{Me}^+(x-\tfrac{1}{2}\mathrm{i}\pi) \\ \text{gives}\quad \mathrm{Ce}^*(x) &= \tfrac{1}{2}\{\mathrm{Me}^*(x)+\mathrm{Me}^*(-x)\}, \\ \mathrm{Se}^*(x) &= -\tfrac{1}{2}\mathrm{i}\{\mathrm{Me}^*(x)-\mathrm{Me}^*(-x)\}. \end{aligned}\right\} \qquad (4.2.11)$$

For $y_1(\pm \mathrm{i}x \pm \frac{1}{2}\pi)$ use the formulae from (4.2.9).

4.3. *Bases and relations*, $\lambda > c_0(q)$

(*a*) *Auxiliary parameters* $(\hat\beta, \Phi)$

The parameter $\hat\beta > 0$, and $\Phi = \frac{1}{2}\arg\beta > 0.$ (4.3.1)

For the determination of the branch of $\arg\beta$ see §3.3 above.

Connection formulae (cf. ((4.1.4), (4.1.5)):

$$y_1(-z) = |\beta|\,\mathrm{e}^{\pm 2\mathrm{i}\Phi}y_1(z) \mp \mathrm{i}\hat\beta y_1(z\pm\pi), \qquad (4.3.2)$$

where
$$|\beta|^2 = 1+\hat\beta^2$$
and
$$\hat\beta\cosh(\pi\mu) = |\beta|\sin 2\Phi.$$

The case of integer order (see §1(*a*)):

$$\left.\begin{aligned} \cosh(\pi\mu) &= (|\beta|/\hat\beta)\sin 2\Phi = e_1, \quad\text{and}\quad |\beta|\cos 2\Phi = e_2, \\ \text{where}\quad e_1, e_2 &= \pm 1, \\ \text{so that}\quad \tan 2\Phi &= e_1e_2\hat\beta; \\ \tan\Phi &= e_1[\beta^*]^{-e_2}, \\ \text{where}\quad \beta^* &= (1+|\beta|)/\hat\beta = \hat\beta/(|\beta|-1). \end{aligned}\right\} \qquad (4.3.3)$$

These four relations follow from (4.3.2), and any one is a necessary and sufficient condition for integer order. Values of e_1, e_2, Φ corresponding to the various periodic Mathieu functions are given in table 2.

TABLE 2. FUNCTIONS OF INTEGER ORDER

$q < 0$	$q > 0$	(e_1, e_2)	$2\Phi\ (0 < \theta = \arctan\hat\beta < \frac{1}{2}\pi)$
ce_{2m}	ce_{2m}	$(+1, -1)$	$(2m+1)\pi-\theta$
ce_{2m+1}	se_{2m+1}	$(-1, -1)$	$(2m+1)\pi+\theta$
se_{2m+1}	ce_{2m+1}	$(-1, +1)$	$(2m+2)\pi-\theta$
se_{2m+2}	se_{2m+2}	$(+1, +1)$	$(2m+2)\pi+\theta$

(*b*) *Modified functions*, $q < 0$; $y''-(\lambda+2h^2\cosh 2x)\,y = 0$

Basis: $\{y_1(\mathrm{i}x), y_1(-\mathrm{i}x)\}.$ (4.3.4)

Definitions and connection formulae

$$\left.\begin{aligned} \left.\begin{aligned}\mathrm{Ce}(x)\\ \mathrm{Se}(x)\end{aligned}\right\} &= \tfrac{1}{2}\{y_1(-\mathrm{i}x)\pm y_1(\mathrm{i}x)\}, \\ \mathrm{Me}(x) &= \{|\beta|\cos 2\Phi+\mathrm{i}\hat\beta\sinh(\pi\mu)\}y_1(\mathrm{i}x)-y_1(-\mathrm{i}x). \end{aligned}\right\} \qquad (4.3.5)$$

For integer order, the last reduces to

$$\mathrm{Me}(x) = e_2y_1(\mathrm{i}x)-y_1(-\mathrm{i}x) = -e_2\,\mathrm{Me}(-x).$$

(*c*) *Ordinary functions*, $q < 0$; $y'' + (\lambda + 2h^2 \cos 2x)\, y = 0$

Basis: on $\{x : 0 \leqslant x \leqslant \frac{1}{2}\pi\}$ $\qquad \{y_3(x),\, y_3(\pi - x)\},$ (4.3.6)

where $\qquad y_3(x) = \frac{1}{2}\{\mathrm{e}^{-\mathrm{i}\Phi} y_1(x) + \mathrm{e}^{\mathrm{i}\Phi} y_1(-x)\}.$

This pair of solutions satisfies the criteria of Miller (1950); see §3.2.

Definitions and connection formulae:

$$\left.\begin{aligned} \mathrm{ce}\,(x) &= \mathrm{Ce}\,(\mathrm{i}x) = \cos\Phi\, y_3(x) + [\beta^*]^{-1} \sin\Phi\, y_3(\pi - x), \\ \mathrm{se}\,(x) &= -\mathrm{i}\, \mathrm{Se}\,(\mathrm{i}x) = \sin\Phi\, y_3(x) - [\beta^*]^{-1} \cos\Phi\, y_3(\pi - x), \\ \mathrm{me}\,(x) &= \{|\beta| \sinh(\pi\mu) + [|\beta|/\hat\beta] \sin 2\Phi\}\, y_3(x) - |\beta| \cos 2\Phi\, y_3(\pi - x), \\ y_1(\pm x) &= \mathrm{e}^{\pm\mathrm{i}\Phi}\{y_3(x) \mp \mathrm{i}[\beta^*]^{-1}\, y_3(\pi - x)\}, \end{aligned}\right\} \tag{4.3.7}$$

where β^* is defined as under (4.3.3).

The formulae for ce (x), se (x) can readily be reverted to give $y_3(x)$, $y_3(\pi - x)$ in terms of ce (x), se (x).

Period relations:

$$\left.\begin{aligned} \hat\beta\, \mathrm{ce}\,(x+\pi) &= |\beta| \sin 2\Phi\, \mathrm{ce}\,(x) + (1 + |\beta| \cos 2\Phi)\, \mathrm{se}\,(x), \\ \hat\beta\, \mathrm{se}\,(x+\pi) &= (1 - |\beta| \cos 2\Phi)\, \mathrm{ce}\,(x) + |\beta| \sin 2\Phi\, \mathrm{se}\,(x). \end{aligned}\right\} \tag{4.3.8}$$

Integer order: $\qquad \mathrm{me}\,(x) = e_1 y_3(x) - e_2 y_3(\pi - x).$

If $e_2 = 1$,

$$\begin{aligned} \mathrm{se}\,(x+\pi) &= e_1\, \mathrm{se}\,(x), \\ \mathrm{ce}\,(x+\pi) &= e_1\, \mathrm{ce}\,(x) + 2\hat\beta^{-1}\, \mathrm{se}\,(x), \\ \mathrm{me}\,(x) &= e_1 \operatorname{cosec}\Phi\, \mathrm{se}\,(x), \end{aligned}$$

and ce (x) is a multiple of ge (x). (4.3.9)

If $e_2 = -1$,

$$\begin{aligned} \mathrm{ce}\,(x+\pi) &= e_1\, \mathrm{ce}\,(x), \\ \mathrm{se}\,(x+\pi) &= e_1\, \mathrm{se}\,(x) + 2\hat\beta^{-1}\, \mathrm{ce}\,(x), \\ \mathrm{me}\,(x) &= e_1 \sec\Phi\, \mathrm{ce}\,(x), \end{aligned}$$

and se (x) is a multiple of fe (x).

Special values:

$$\left.\begin{aligned} y_1(0) &= \mathrm{ce}\,(0), & y_1'(0) &= \mathrm{i}\, \mathrm{se}'\,(0), \\ y_3(0) &= \cos\Phi\, \mathrm{ce}\,(0), & y_3'(0) &= \sin\Phi\, \mathrm{se}'\,(0), \\ y_3(\pi) &= \beta^* \sin\Phi\, \mathrm{ce}\,(0), & y_3'(\pi) &= \beta^* \cos\Phi\, \mathrm{se}'\,(0). \end{aligned}\right\} \tag{4.3.10}$$

(*d*) *Ordinary functions*, $q > 0$; $y'' + (\lambda - 2h^2 \cos 2x)\, y = 0$

Basis: $\qquad \{y_3(\frac{1}{2}\pi + x),\, y_3(\frac{1}{2}\pi - x)\}.$ (4.3.11)

Definitions and connection formulae:

$$\left.\begin{aligned} \left.\begin{aligned} \mathrm{ce}^*\,(x) \\ \mathrm{se}^*\,(x) \end{aligned}\right\} &= \tfrac{1}{2}\{y_3(\tfrac{1}{2}\pi + x) \pm y_3(\tfrac{1}{2}\pi - x)\}, \\ \mathrm{me}^*\,(x) &= \mathrm{me}\,(\tfrac{1}{2}\pi \pm x), \\ y_1(\pm\tfrac{1}{2}\pi \pm x) &= \mathrm{e}^{\pm\mathrm{i}\Phi}\{y_3(\tfrac{1}{2}\pi + x) \mp \mathrm{i}[\beta^*]^{-1}\, y_3(\tfrac{1}{2}\pi - x)\}. \end{aligned}\right\} \tag{4.3.12}$$

Period relations:

$$\left.\begin{aligned} \mathrm{ce}^*\,(x+\pi) &= |\beta|\hat\beta^{-1} \sin 2\Phi\, \mathrm{ce}^*\,(x) + (\hat\beta^{-1} \sin 2\Phi + \cos 2\Phi)\, \mathrm{se}^*\,(x), \\ \mathrm{se}^*\,(x+\pi) &= (\hat\beta^{-1} \sin 2\Phi - \cos 2\Phi)\, \mathrm{ce}^*\,(x) + |\beta|\hat\beta^{-1} \sin 2\Phi\, \mathrm{se}^*\,(x). \end{aligned}\right\} \tag{4.3.13}$$

Special values:

$$\left.\begin{aligned} y_3(\tfrac{1}{2}\pi) &= \mathrm{ce}^*\,(0), \quad y_3'(\tfrac{1}{2}\pi) = \mathrm{se}^{*\prime}\,(0),\\ y_1(\pm\tfrac{1}{2}\pi) &= \mathrm{e}^{\pm \mathrm{i}\Phi}\{1\mp \mathrm{i}[\beta^*]^{-1}\}\,\mathrm{ce}^*\,(0), \quad y_1'(\pm\tfrac{1}{2}\pi) = \mathrm{e}^{\pm\mathrm{i}\Phi}\{1\pm \mathrm{i}[\beta^*]^{-1}\}\,\mathrm{se}^{*\prime}\,(0).\end{aligned}\right\} \tag{4.3.14}$$

Integer order:

$$\left.\begin{aligned} &\text{If } e_1e_2 = 1, & \mathrm{se}^*(x+\pi) &= e_1\,\mathrm{se}^*\,(x),\\ & & \mathrm{ce}^*\,(x+\pi) &= e_1\{\mathrm{ce}^*\,(x)+2|\beta|^{-1}\,\mathrm{se}^*\,(x)\},\\ & & \mathrm{me}\,(x) &= 2\,\mathrm{se}^*\,(x).\\ &\text{If } e_1e_2 = -1 & \mathrm{ce}^*(x+\pi) &= e_1\,\mathrm{ce}^*\,(x),\\ & & \mathrm{se}^*\,(x+\pi) &= e_1\{\mathrm{se}^*\,(x)+2|\beta|^{-1}\,\mathrm{ce}^*\,(x)\},\\ & & \mathrm{me}(x) &= 2\,\mathrm{ce}^*\,(x).\end{aligned}\right\} \tag{4.3.15}$$

Relations between periodic solutions in the cases $q<0$, $q>0$*:*

$e_2 \backslash e_1$	$+1$	-1
$+1$	$\mathrm{se}\,(x+\tfrac{1}{2}\pi) = 2\sin\Phi\,\mathrm{se}^*\,(x)$	$\mathrm{se}\,(x+\tfrac{1}{2}\pi) = 2\sin\Phi\,\mathrm{ce}^*\,(x)$
-1	$\mathrm{ce}\,(x+\tfrac{1}{2}\pi) = 2\cos\Phi\,\mathrm{ce}^*\,(x)$	$\mathrm{ce}\,(x+\tfrac{1}{2}\pi) = 2\cos\Phi\,\mathrm{se}^*\,(x)$

(4.3.16)

(*e*) *Modified functions*, $q>0$; $y''-(\lambda-2h^2\cosh 2x)\,y = 0$

Basis:

$$\{y_4(x),\,y_4(-x)\}, \tag{4.3.17}$$

where

$$y_4(x) = \tfrac{1}{2}\{\mathrm{e}^{\mathrm{i}(\frac{1}{4}\pi-\Phi)}\,y_1(\mathrm{i}x+\tfrac{1}{2}\pi)+\mathrm{e}^{-\mathrm{i}(\frac{1}{4}\pi-\Phi)}y_1(\mathrm{i}x-\tfrac{1}{2}\pi)\}.$$

This pair of solutions satisfies the criteria of Miller (1950); see §3.2.

Definitions and connection formulae:

$$\left.\begin{aligned} &\left.\begin{aligned}\mathrm{Ce}^*\,(x) &= \mathrm{ce}^*(\mathrm{i}x)\\ \mathrm{Se}^*\,(x) &= -\mathrm{i}\,\mathrm{se}^*\,(\mathrm{i}x)\end{aligned}\right\} = 2^{-\frac{1}{2}}[1+|\beta|-\hat{\beta}]\,\{y_4(-x)\pm y_4(x)\},\\ &\mathrm{Me}^*\,(x) = \mathrm{me}^*\,(\mathrm{i}x) = \{\sin\,(\pi\nu)+\cos 2\Phi\}\,y_4(x)-\hat{\beta}^{-1}\sin 2\Phi\,y_4(-x),\\ &y_1(\mathrm{i}x\pm\tfrac{1}{2}\pi) = \mathrm{e}^{\pm\mathrm{i}(\Phi-\frac{1}{4}\pi)}\{y_4(x)\pm\mathrm{i}[|\beta|+\hat{\beta}]^{-1}\,y_4(-x)\}.\end{aligned}\right\} \tag{4.3.18}$$

Integer order:

The modified function corresponding to the periodic ordinary function satisfies

$e_1 \backslash e_2$		$+1$	-1
$+1$	$\mathrm{Im}\,[\hat{\beta}y_1(\tfrac{1}{2}\pi+\mathrm{i}x)] =$	$-2\sin\Phi\,\mathrm{Se}^*\,(x)$	$-2\cos\Phi\,\mathrm{Ce}^*\,(x)$
-1	$\mathrm{Re}\,[\hat{\beta}y_1(\tfrac{1}{2}\pi+\mathrm{i}x)] =$	$-2\sin\Phi\,\mathrm{Ce}^*\,(x)$	$-2\cos\Phi\,\mathrm{Se}^*\,(x)$

(4.3.19)

For other solutions see §3.4*b* above.

References (Part II)

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